Journal of Statistical Physics, Vol. 99, Nos. 3/4, 2000

Soliton Solutions of Integrable Hierarchies and Coulomb Plasmas

Igor Loutsenko[1] **and Vyacheslav Spiridonov**[2]

Received April 13, 1999; final December 20, 1999

Some direct relations are given between soliton solutions of integrable hierarchies and thermodynamic quantities of the Coulomb plasmas on the plane. We find that certain soliton solutions of the Kadomtsev–Petviashvili (KP) and B-type KP (BKP) hierarchies describe 2D one- or two-component lattice plasmas at special boundary conditions and fixed temperatures. It is shown that different reductions of integrable hierarchies describe pure or dipole Coulomb gases on 1D submanifolds embedded in the 2D space.

KEY WORDS: Coulomb plasmas; integrable hierarchies; tau functions; solitons; dipole gases.

1. INTRODUCTION

Recently we have shown[(1)] that the grand partition functions of some 1D lattice gas models or equivalent to them partition functions of some Ising chains, coincide with the N-soliton tau functions of various hierarchies of integrable nonlinear evolution equations. The present paper contains a detailed comparison of exactly solvable Coulomb plasma models on lattices with integrable equations. We discuss statistical mechanics of the Coulomb (logarithmic interaction) gases on intrinsic 2D geometric figures and various 1D submanifolds of the plane. In this way we do not merely reinterpret previously known results,[(2–7)] but also reveal a number of new exactly solvable (at fixed temperatures) plasma models. It is natural to

[1] CRM, Université de Montréal, C.P. 6128, Succ. Centre-Ville, Montréal, Quebéc, Canada.
[2] Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, Moscow Region 141980, Russia; e-mail: sup@thsun1.jinr.ru.

expect that the connection with integrable equations gives a clue to the classification of such models.

A classical Coulomb plasma is a system of charged particles interacting through the Coulomb potential. On the plane this potential is defined as a solution to the Poisson equation $\Delta V(r, r') = -2\pi\delta(r - r')$, with certain boundary conditions. For the plane without boundaries this equation has the solution $V(z, z') = -\ln |z - z'|$, where $z = x + \imath y$. A system of particles forms a stable plasma if the opposite valued charges do not recombine with each other forming a gas of neutral molecules. In general the location of plasma is constrained to some domain in which case there is a nontrivial interaction with the boundaries of this domain. In particular, the normal component of the electric field $\mathscr{E} = -\nabla V$ should vanish on the surface of an ideal dielectric, $\mathscr{E}_n = 0$, while the tangent component of this field vanishes on the surface of an ideal conductor, $\mathscr{E}_t = 0$. A useful way of solving the Poisson equation with such boundary conditions is provided by the method of images. In the present work we consider systems where every charge has either a finite number of images created by the boundaries or boundary conditions create periodic lattices of images. For a finite number of images solution of the Poisson equation is given by a finite sum of logarithmic potentials created by a charge and its images.

The energy of such systems of N particles has the following form

$$E_N = \sum_{1 \leqslant i < j \leqslant N} Q_i Q_j V(z_i, z_j) + \sum_{1 \leqslant i \leqslant N} Q_i^2 v(z_i) + \sum_{1 \leqslant i \leqslant N} Q_i \Phi(z_i) \tag{1}$$

where $z_i = x_i + \imath y_i$ and Q_i are the coordinates and charges of the particles on the plane. The first term in (1) is the energy of interaction between different charges. The second terms is the sum of self-energies. In general it arises as the energy of interaction between a charge and its own images. The third term describes an interaction of charges with external fields.

The grand partition function of a system of particles of s different species is

$$\begin{aligned} G &= \sum_{n_1=1}^{N_1} \cdots \sum_{n_s=1}^{N_s} \frac{\zeta_1^{n_1} \cdots \zeta_s^{n_s}}{n_1! \cdots n_s!} Z_{n_1 \cdots n_s} \\ &= \sum_{n_1=1}^{N_1} \cdots \sum_{n_s=1}^{N_s} \frac{1}{n_1! \cdots n_s!} \int e^{-\Gamma H_{n_1 \cdots n_s} + \mu_1 n_1 + \cdots + \mu_s n_s} \, d\Omega \end{aligned}$$

where $\mu_1, \ldots, \mu_s$ denote chemical potentials ($\zeta_s = e^{\mu_s}$ are the fugacities) and Ω is an integration measure over the configuration space occupied by particles. Here we have introduced the dimensionless inverse temperature $\Gamma = \beta Q^2$, $\beta = 1/kT$, and the dimensionless Hamiltonian

$$H_{n_1 \cdots n_s} = \frac{1}{2} \sum_{i \neq j} q_i q_j V(z_i, z_j) + \sum_i q_i^2 v(z_i) + \sum_i q_i \phi(z_i) \tag{2}$$

where $q_i = Q_i/Q$ and $\phi(z) = \Phi(z)/Q$. For one and two component plasmas it is convenient to choose $Q = |Q_i|$, so that the dimensionless charges $q_i = 1$ in the one component case and $q_i = \pm 1$ in the two component case.

The 2D one and two component plasma models have been solved for a variety of boundary conditions at a special value of the inverse temperature $\Gamma = 2$ (see, e.g., refs. 2–7 and references therein). In technical terms the solution is possible due to different determinant representations (the Cauchy determinant for the two component plasma and the Vandermonde determinant for the one component case). Such representations allow one to solve models of log-gases on a line with the transverse boundary conditions (e.g., see ref. 7). Note that the majority of previous works is devoted to the continuous space models, for an account of the lattice cases see, e.g., refs. 1, 2, 6. The literature on the Coulomb gases is enormous, many statistical mechanics models have been related to them,[(8)] there is a relation to conformal field theories, etc. Still, the identification of Coulomb plasmas on lattices with some boundaries and of the famous multisoliton systems has been missed in the previous investigations.

2. BASIC OBSERVATIONS

Let us consider a Coulomb plasma on a lattice. We suppose that each type of particles can occupy only a discrete set of points in the complex plane. E.g., in the two component case the positive and negative charge particles occupy sublattices $\{z_+\}$ and $\{z_-\}$. We denote the union of all sublattices as $\{z\}$. No more than one particle is allowed at each site. In this case, the integrals over Ω are replaced by discrete sums over the lattice points and the whole partition function can be rewritten in the following form[(9)]

$$G = \sum_{\{\sigma\}} \exp\left(\frac{1}{2} \sum_{z \neq z'} W(z, z')\, \sigma(z)\, \sigma(z') + \sum_{\{z\}} w(z)\, \sigma(z) \right) \tag{3}$$

$$W(z, z') = -\Gamma q(z)\, q(z')\, V(z, z'), \qquad w(z) = \mu(z) - \Gamma (q^2(z)\, v(z) + q(z)\, \phi(z)) \tag{4}$$

and $\sigma(z)=0$ or 1 is an occupation number of the site with the coordinate z. The variables $q(z)$, $\mu(z)$ are some functions of the lattice coordinates. For example, $q(z_\pm)=\pm 1$, $\mu(z_\pm)=\mu_\pm$ for the two component plasma.

Now, let us write out the τ-function of N-soliton solutions of some integrable hierarchy in the Hirota form[10]

$$\tau_N=\sum_{\sigma=0,1} \exp\left(\frac{1}{2}\sum_{z\neq z'} A_{zz'}\sigma(z)\,\sigma(z')+\sum_{\{z\}}\theta(z)\,\sigma(z)\right) \tag{5}$$

where the variable z takes N discrete values describing spectral characteristics of solitons. The function $\theta(z)$ parameterizes phases of solitons with the index z and $A_{zz'}$ is the phase shift acquired as a result of the scattering of solitons with the indices z and z' off each other.

Evidently, the expressions (3) and (5) have the same form. One just needs to make proper identifications between the phase shifts $A_{zz'}$ and the interaction potentials $W(z, z')$, and between the phases $\theta(z)$ and the function $w(z)$. Such an identification appears to be valid for the celebrated Korteweg–de Vries (KdV) equation and higher order members of the KdV-hierarchy, Kadomtsev–Petviashvili (KP) hierarchy and its B-type reduction, and some other integrable equations.[10] In the next section we consider in detail an identification of the KP hierarchy solitons and plasma particles.

Before passing to that let us remark that lattice gas models are related to the Ising models.[9] Indeed, substituting $\sigma(z)=(s(z)+1)/2$, $s(z)=\pm 1$, into (3), we get a formula for the partition function of an Ising model up to some constant multiplicative factor

$$G=\sum_{\{s\}} \exp\left(\frac{1}{2}\sum_{z\neq z'} J(z,z')\,s(z)\,s(z')+\sum_{\{z\}} H(z)\,s(z)\right) \tag{6}$$

where the exchange constants $J(z, z')$ and the magnetic field $H(z)$ have the form (we have absorbed the variable Γ into their definition)

$$J(z,z')=\frac{1}{4}W(z,z'),\qquad H(z)=\frac{1}{2}w(z)+\frac{1}{4}\sum_{z',\, z'\neq z} W(z,z') \tag{7}$$

Technically, it appears that the Ising representation of the lattice plasma grand partition function is more convenient for writing it in the determinant form.

3. KP HIERARCHY

Multisoliton solution of the KP-hierarchy is presentable in the form (5) with the following parameterization of the soliton phases and phase shifts (we borrow this expression from ref. 11 where an explicit form of the equations themselves can be found as well):

$$A_{zz'} = \ln \frac{(a_z - a_{z'})(b_z - b_{z'})}{(a_s + b_{z'})(b_z + a_{z'})}, \qquad \theta(z) = \theta^{(0)}(z) + \sum_{p=1}^{\infty} (a_z^p - (-b_z)^p)\, t_p \tag{8}$$

where t_p is the pth KP "time" and a_z, b_z are some arbitrary functions of z. If we take

$$a_z = z = x + \imath y, \qquad b_z = -z^* - x + \imath y, \qquad y \geqslant 0$$

then $A_{zz'} = W(z, z') = -2V(z, z')$, where

$$V(z, z') = -\ln |z - z'| + \ln |z^* - z'| \tag{9}$$

is the potential at the point z created by a positive unit charge particle places at the point z' over the conducting surface occupying the $y \leqslant 0$ region ($V(z, z')$ solves the Poisson equation with the tangent boundary condition $\mathscr{E}_x(y=0) = 0$). Equally, one may say that this is an effective potential created by a positive charge at the point z', $\Im z' > 0$, and its image of opposite charge located at $(z')^*$. Comparing (9) with the original definition (4) one finds that the temperature is fixed and $\Gamma = 2$. Thus a particular KP N-soliton solution corresponds to the Coulomb plasma in the upper half plane $\Im z > 0$ with metallic boundary along the x-axis. The situation is depicted in the Fig. 1.

Let us shift $z \to z + \imath a$, a real and take the limit $a \to \infty$, i.e., take the plasma far away from the boundary. This leads to some divergences in the energy which can be removed by addition of an appropriate diverging constant to the initial Hamiltonian. As a result one gets the pure plasma system at the inverse temperature $\Gamma = 2$. This value corresponds to the standard normalization in the random matrix theory.[2] The normalization of the temperature in our previous papers[1] was chosen as $\Gamma = 1$ since there we were discussing Ising chains without detailed comparison with the Coulomb systems which is a goal of the present work.

The identification $w(z) = \theta(z)$ allows one to write the following expression for the zero-time phases $\theta^{(0)}(z)$ in (8)

$$\theta^{(0)}(z) = \mu - \Gamma(\ln |z^* - z| + \phi(z)), \qquad \Gamma = 2 \tag{10}$$

where the second term corresponds to the "charge-image" interaction, and the last term describes the potential of the field created by a neutralizing background of the density $\rho(z)$:

$$\Delta\phi(z) = -2\pi\rho(z), \qquad \phi_z(y=0) = 0 \tag{11}$$

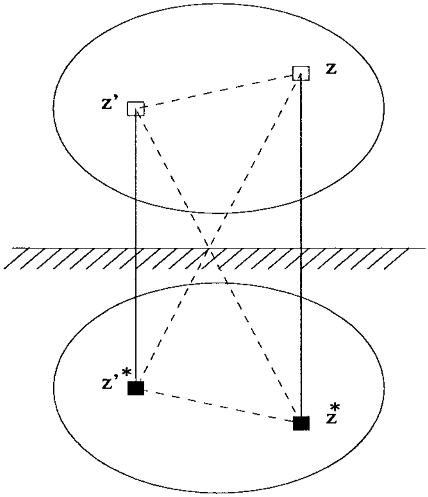

Fig. 1. KP equation: one component two-dimensional plasma above an ideal conductor. Positive charges are shown as white squares while their negative images are show as black squares. Interactions between different charges are shown by dashed lines, while the interactions between charges and their own images are shown by the solid lines.

Contributions from the KP "times" $\sum_{p=0}^{\infty} (z^p - (z^*)^p)\, t_p = -\Gamma \phi_{\text{ext}}(z)$ correspond to an external electric field. Since the Laplacian of this part is zero the corresponding density of charges is zero, i.e., this field is generated by external distant charges. For instance, the contribution of the first time $(z - z^*)\, t_1$ corresponds to the homogeneous electric field perpendicular to the boundary. For reality of the potential the time t_1 has to be purely imaginary.

The system of distant charges g_i located at the points w_i above the conductor create the following electrostatic potential at the point z

$$\phi_{\text{ext}}(z) = -\sum_i g_i \ln \frac{|z - w_i|}{|z - w_i^*|}$$

Since the external charges are fare from the origin $|z| \ll |w_i|$, we can expand the potential $\phi_{\text{ext}}(z)$ in the Taylor series and get as KP times

$$t_p \equiv \frac{\Gamma}{2p} \sum_i g_i \left(\frac{1}{(w_i^*)^p} - \frac{1}{w_i^p} \right)$$

In this picture the KP times take imaginary values automatically. One may conclude that the general imaginary times evolution of a special system of

KP hierarchy solitons describes electrostatics of a plasma in a varying external electric field.

Using the standard scaling arguments one can see that for the continuous space version of the present model the average pressure changes the sign at the inverse temperature $\Gamma = 2$. As a result for $\Gamma \geqslant 2$ the particles stick to the surface of the conductor and the system collapses. The situation is cured if the plasma is confined to a domain which does not touch the boundary. Such an unphysical behavior is avoided in the lattice version of the model (the minimal lattice spacing may be interpreted as the radius of a hard core repulsive interaction). Connection with the KP solitons takes place exactly at $\Gamma = 2$.

The Poisson equation and the boundary conditions $\mathscr{E}_n = 0$ or $\mathscr{E}_t = 0$ are satisfied for an appropriate conformal change of the variable $z \to f(z)$. For instance, choosing soliton parameters in (8) as $a_z = z^2$, $b_z = -(z^*)^2$, we obtain

$$W(z, z') = 2 \ln \left| \frac{z^2 - (z')^2}{(z^*)^2 - (z')^2} \right|$$

corresponding to the interaction of two charged particles in the rectangular corner with metallic walls along the x and y axes. Higher degree monomial maps $z \to z^n$ put the plasma into a corner with the π/n angle between the conducting walls.

The exponential map $a_z = \exp(\pi z/L)$, $b_z = -\exp(\pi z^*/L)$, generates the W-potential

$$W(z, z') = 2 \ln \left| \frac{\sinh(\pi/2L)(z - z')}{\sinh(\pi/2L)(z^* - z')} \right| \tag{12}$$

which describes the plasma in the strip $\Im z = (0, L)$ between two parallel conductors.

The choice $a_z = \exp(-\pi x/L)$, $b_z = -\exp(-\pi(x + \alpha)/L)$, results in

$$W(x, x') = \ln \frac{\sinh^2(\pi/2L)(x - x')}{\sinh(\pi/2L)(x - x' - \alpha) \sinh(\pi/2L)(x - x' + \alpha)} \tag{13}$$

Solution of the Poisson equation with periodic boundary condition along the y-axis with the period $2L$ is given by the potential[7]

$$V(z, z') = -\ln \left| \sinh \frac{\pi}{2L} (z - z') \right|$$

Therefore one can interpret (13) as the interaction energy of two neutral dipoles in the periodic background with the distance between charges in the molecule equal to α (the internal energy of dipoles is neglected). These dipoles all lie on the x-axis and have identical orientation. Since $W = -\Gamma V$, we see that the effective inverse temperature is twice smaller than in the previous cases, i.e., $\Gamma = 1$.

Let $a_z = \mathrm{sn}^2\, z$, $b_z = -\mathrm{sn}^2\, z^*$, where sn z is the Jacobian elliptic function with the periods L_x, $\imath L_y$. Then one gets the W-potential

$$W(z, z') = 2 \ln \left| \frac{\mathrm{sn}^2\, z - \mathrm{sn}^2\, z'}{\mathrm{sn}^2\, z^* - \mathrm{sn}^2\, z'} \right| = 2 \ln \left| \frac{\theta_1(z - z')\, \theta_1(z + z')}{\theta_1(z^* - z')\, \theta_1(z^* + z')} \right| \tag{14}$$

where $\theta_1(z)$ is the Jacobi theta-function. It vanishes when z lies on the boundary of a $L_x \times L_y$ rectangle, i.e., we have the plasma in a box with the conducting walls ($\Gamma = 2$). For small z, z' (or for the large size box $L_x, L_y \to \infty$) one recovers plasma in the rectangular corner.

The choice $a_z = b_z$ corresponds to the reduction of KP to KdV equation. However, the potential W is real in this case only for purely real or imaginary z. Otherwise one gets complex tau function and there is no complete coincidence of (3) and (5)—G is real due to the module signs in the Coulomb potential. One may try to identify complex multisoliton solutions with a special system of interacting electric and magnetic charges.[(8)] We do not consider such possibilities and limit ourselves to the purely electric systems with real interaction energy.

Consider the situation when lattice points $\{z\}$ consists of two subsets $\{z_\pm\}$, $\{z\} = \{z_-\} \cup \{z_+\}$. Choosing the following identification of parameters in (8)

$$a_{z_-} = z_-, \qquad b_{z_-} = -z_-^*, \qquad a_{z_+} = z_+^*, \qquad b_{z_+} = -z_+ \tag{15}$$

we get a model of the two component plasma above an ideal conductor

$$W(z_\pm, z'_\pm) = 2 \ln |z_\pm - z'_\pm| - 2 \ln |z_\pm^* - z'_\pm|$$

$$W(z_\pm, z'_\mp) = 2 \ln |z_\pm^* - z'_\mp| - 2 \ln |z_\pm - z'_\mp|$$

$$\theta(z_\pm) = \mu_\pm - 2 \ln |z_\pm - z_\pm^*| \mp 2\phi(z_\pm)$$

The inverse temperature is again $\Gamma = 2$. After the conformal transformations $z \to z^2$, e^z, etc. one gets the two-component plasma in the metallic rectangular corner, a strip, etc.

One component plasma is not stable without metallic boundary or neutralizing background, since its particles tend toward boundaries repelling each other with long range forces. In the two component case screening is possible if the system is neutral and the temperature is high enough. Simple scaling analysis shows that the pure two component plasma undergoes a transition to neutral molecules in the bulk at $\Gamma = 2$.[2] Again, there is no such a transition in the lattice case because of the hard-core repulsion.

It is well known that the two component homogeneous plasma without boundaries is a specific representation of the quantum Sine-Gordon or Thirring model.[12] The inverse temperature $\Gamma = 2$ corresponds to the free fermion point, but the model is (in principle) integrable for any Γ. The Sine-Gordon model requires renormalization if the coupling constant exceeds a critical value. The lattice spacing is equivalent to a cutoff in the corresponding field theory. Thus it is natural to associate the two component lattice plasma above the metallic boundary with some discretized boundary Sine-Gordon model.

4. KP HIERARCHY: SOME REDUCTIONS

In this section we consider a number of one-dimensional reductions of the KP hierarchy and some self-similar soliton solutions. We describe here only the most popular integrable systems and do not cover all possible cases and their Coulomb gas interpretations.

1. We begin with the reduction considered earlier in the literature.[2] In this case, plasma is restricted to the line $y = Y$. The W-potential is translationally invariant and equals to

$$W(x, x') = \ln \frac{(x - x')^2}{Y^2 + (x - x')^2}$$

Interaction energies of charges with their own images are constant and may be neglected.

2. Reduction to the KdV hierarchy. In this case particles move along the vertical line $x = 0$ and the W-potential is

$$W(y, y') = 2 \ln \left| \frac{y - y'}{y + y'} \right| \tag{16}$$

Now a non-trivial self-interaction term $\propto \ln |2y|$ enters the definition of soliton phases.

3. Discrete KdV hierarchy. The phase shifts have the form[10]

$$A_{zz'} = \ln \frac{\sinh^2(z - z')/2}{\sinh^2(z + z')/2}$$

Taking z to be purely imaginary $z = \imath\alpha$ we get the following result

$$W(\alpha, \alpha') = 2 \ln \left| \frac{\sin(\alpha - \alpha')/2}{\sin(\alpha + \alpha')/2} \right|$$

which corresponds to the plasma restricted to an arc with the center of the corresponding circle at the conductor's surface. The self-interaction energy is $-\ln |2 \sin \alpha|$. It is sufficient to put $z = e^{\imath\alpha}$ in (9) in order to get this system from the KP soliton solutions.

4. The reductions admitting both left and right moving solitons correspond to the plasma restricted to domains of disjoint parts. E.g., the Toda lattice case[10]

$$A_{zz'} = 2 \ln \frac{\varepsilon(z)\, z - \varepsilon(z')\, z'}{1 - \varepsilon(z)\, \varepsilon(z')\, zz'}, \qquad \varepsilon(z) = \pm 1$$

corresponds for $z = e^{\imath\alpha}$ to the two component plasma on the half circle with the center lying upon the conductor surface. The positive charge particles occupy the subsector $\alpha \in [0, \pi/2]$ and the negative charge particles are located in the subsector $\alpha[\pi/2, \pi]$.

5. The Boussinesq equation[13] corresponds to a one component plasma on disjoint halves of the hyperbola $3x^2 = y^2 + 1$ situated above the conductor occupying the $y \leqslant 0$ half plane:

$$A_{yy'} = \ln \frac{(\varepsilon(y)\, x(y) - \varepsilon(y')\, x(y'))^2 + (y - y')^2}{(\varepsilon(y)\, x(y) - \varepsilon(y')\, x(y'))^2 + (y + y')^2}$$

$$x(y) = \sqrt{\frac{y^2 + 1}{3}}, \qquad \varepsilon(y) = \pm 1$$

It would be interesting to find the equation whose soliton phase shifts are equal to (14).

Consider some specific lattice plasma configurations associated with self-similar soliton solutions of integrable equations. Take first the KP situation (12) and restrict the corresponding plasma to a line parallel to the conductor surfaces $\Im z = Y$. Applying this constraint, we get

$$W(x - x') = -\ln \left(\sin^2 \frac{\pi Y}{L} \coth^2 \frac{\pi(x - x')}{2L} + \cos^2 \frac{\pi Y}{L} \right)$$

The simplest self-similar reduction corresponds to the homogeneous one-dimensional lattice parallel to the x-axis. Then the nth charge has the coordinates $z_n = X + hn + \imath Y$, where X is some fixed constant and h is the lattice spacing. In this case soliton momenta a_z and b_z form one geometric progression. The simplest KdV self-similar reduction corresponds to the case when the line is located at equal distances between parallel conductors $Y = L/2$:

$$W(x - x') = \ln \tanh^2 \frac{\pi(x - x')}{2L} \tag{17}$$

The M-periodic reduction corresponds to the situation when soliton momenta are composed from M distinct geometric progressions. In this case the lattice $\{z\}$ consists of M homogeneous sublattices. In the plasma language it corresponds to the model where plasma moves on M distinct parallel lines between two conductors with the coordinates $\{z\} = \{\imath Y_p + X_p + nh,\ p = 1,\ldots, M,\ n = 0, 1,\ldots\}$. If one sets $Y_p = L/2$ then all these sublattices are situated upon the middle line and this case corresponds to the general self-similar KdV soliton potentials of ref. 14.

In a similar way one can place plasma upon M parallel lines with fixed x-coordinates, $x = X_p$, which are situated above the $y \leqslant 0$ conductor. In this case self-similar lattices are described by restriction of y to the union of M geometric progressions $\{z\} = \{X_p + \imath Y_p q^n,\ q < 1\}$. However, such configurations are not safe from the collapse of particles on the walls. The KP self-similar systems are richer than the ones of the KdV or BKP (to be considered below) equations due to the presence of non-trivial translational parts X_p (or Y_p) in the parameterization of the corresponding soliton spectral data a_z, b_z.

Note that the Ising models emerging in this formalism look natural only in the reduced 1D case, when there are no restrictions upon the values of spins at different points. In the 2D picture with boundaries one has to assume that the configuration of spins satisfies some geometric constraints which do not have natural meaning similar to the one existing in the electrostatics. This leads to exchanges which depend not only on the distance between the spins but on the distance to the boundaries as well.

5. BKP HIERARCHY

Multisoliton solution of the B-type KP(BKP) hierarchy has the same form (5) with the following phase shifts and soliton phases (see ref. 11 for these items and the equations themselves):

$$A_{zz'} = \ln \frac{(a_z - a_{z'})(b_z - b_{z'})(a_z - b_{z'})(b_z - a_{a'})}{(a_z + a_{z'})(b_z + b_{z'})(a_z + b_{z'})(b_z + a_{z'})}$$
$$\theta(z) = \theta^{(0)}(z) + \sum_{p=1}^{\infty} (a_z^{2p-1} + b_z^{2p-1})\, t_{2p-1} \tag{18}$$

where t_{2p-1} are the BKP "times." If we take one component plasma and fix $a_z = z = x + \imath y$, $b_z = z^*$, $x > 0$, $y \geqslant 0$, then $A_{zz'} = W(z, z') = -2V(z, z')$, where

$$V(z, z') = -\ln |z - z'| - \ln |z^* - z'| + \ln |z + z'| + \ln |z^* + z'|$$

is the total interaction energy between unit charge particles at the points z, $\Im z > 0$, $\Re z > 0$, and z', $\Im z' > 0$, $\Re z' > 0$, in a domain of the upper right quarter of the plane with an ideal dielectric boundary along the x-axis and an ideal conductor wall along the y-axis. This situation is depicted in the Fig. 2. Comparing with the definition (4) we see that the inverse temperature is fixed and equals to $\Gamma = 2$.

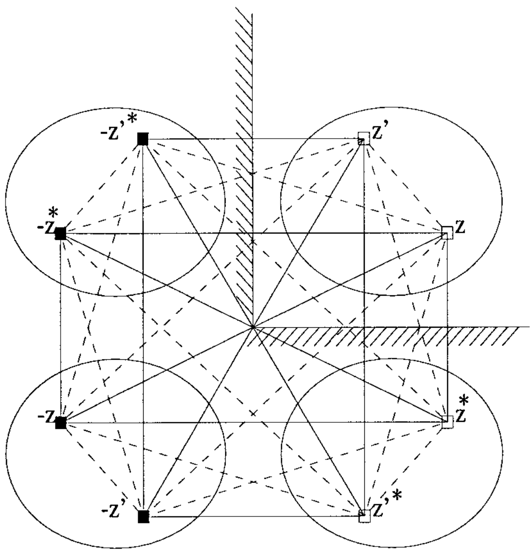

Fig. 2. BKP equation: a one component two-dimensional plasma in the corner between an ideal dielectric (the horizontal axis) and an ideal conductor (the vertical axis). Positive charges are shown as white squares while their negative charge images are shown as the black squares. Interactions between different charges are shown by dashed lines, while the self-interactions between charges and their own images are shown by the solid lines.

If the plasma is taken far away from the corner by appropriate translations, then one gets the pure plasma systems at the effective temperature $\Gamma = 2$. Sliding along the y-axis to infinity one comes to the previously considered plasma associated with the KP equation. Sliding along the x-axis requires a renormalization of the zero energy level after which one gets a plasma above the surface of a dielectric. If we place charges upon the $y = 0$ axis, then the dielectric boundary condition disappears and we get

$$W(x, x') = 4 \ln \left| \frac{x - x'}{x + x'} \right|$$

which corresponds to the plasma model induced by the KdV equation at the inverse temperature $\Gamma = 4$, which is twice higher than in the appropriate KP reduction case.

Identification of the initial phase $\theta^{(0)}(z)$ in (8) is as follows

$$\theta^{(0)}(z) = 2 \ln |z^* - z| - 2 \ln |z^* + z| - 2 \ln |2z| + \mu - 2\phi(z)$$

where the meaning of all terms is similar to that of (10), (11). Contribution of the BKP "times" $\sum_{p=1}^{\infty} (z^{2p-1} + (z^*)^{2p-1})\, t_{2p-1}$ corresponds to an electric field created by external charges satisfying the appropriate boundary conditions.

In complete parallel with the KP case one can consider conformal transformations $z \to z^n$, e^z, etc. and map plasma to various geometric configurations. Considering systems with two sublattices like (15) one arrives at models of two component plasma in the metal-dielectric corner or other bounded regions.

Consider real self-similar reductions of the BKP hierarchy corresponding to dipole gases on a line between two conductors. Choosing $a_i = \exp(-\pi h i/L)$, $b_i = -\exp(-\pi(hi + \alpha)/L)$, $i = 1,..., N$, we get an expression for the grand partition function of the form (3) for a homogeneous lattice gas (here we simply identify ith particle coordinate with i). The particles of this gas interact via the following W-potential

$$W_d(i-j) = W(h(i-j)) - \tfrac{1}{2} W(h(i-j) - \alpha) - \tfrac{1}{2} W(h(i-j) + \alpha) \qquad (19)$$

with W given by (17). It is seen immediately that this is the potential of two dipole molecules consisting of two opposite charges situated at the distance α from each other without the part describing interaction of charges inside the dipoles (which is constant). The distance between molecules at ith and jth site is $h(i-j)$, where h denotes the lattice spacing. The dipoles move along the line situated at equal distances $1/2L$ from two parallel conductors (see Fig. 3). All dipoles are oriented in one direction. Due to the

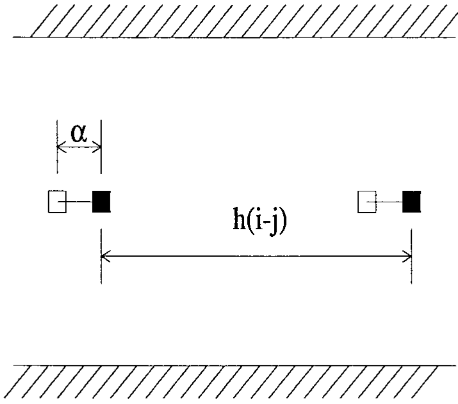

Fig. 3. BKP: dipole gas on the line between two ideal conductors. The dipoles are neutral, oriented identically and lie upon the homogeneous lattice.

dipole gas interpretation one has the inverse temperature $\Gamma = 1$—this is a demonstration of some discrete temperature renormalization effect.

If one takes a_i as above but changes the sign of b_i (i.e., shifts $\alpha \to \alpha + \imath L$), then the signs of the second and third terms in (19) are changed too. This situation corresponds to a dipole gas on the middle line, the dipoles being charged molecules of total charge $+2$ with the same distance between charges α. These molecules are positioned similar to the previous case.

Another lattice gas model of charged dipoles appears if one replaces α in (19) by $\imath(\alpha + L)$, $0 < \alpha < L$. In this case charged dipoles are positioned vertically and symmetrically with respect to the middle line $y = L/2$. The case when α in (19) is replaced by $\imath\alpha$, $0 < \alpha < L$, corresponds to the neutral dipoles gas in the strip between *dielectric* walls. The dipoles are perpendicular to the middle line $y = L/2$, similar to the previous case, and have an identical orientation.

The M-periodic self-similar reductions,[1, 14] when $a_{i+M} = qa_i$, $b_{i+M} = qb_i$, describe the gas consisting of M different types of dipoles. In some particular cases this leads to dipole gases with different orientations of neutral or changed molecules in a strip between the conducting walls. For instance, for the choice

$$a_{2i} = e^{-(\pi/L)(2ih - \alpha/2)}, \qquad b_{2i} = -e^{-(\pi/L)(2ih + \alpha/2)}$$

$$a_{2i+1} = e^{-(\pi/L)((2i+1)h + \alpha/2)}, \qquad b_{2i+1} = e^{-(\pi/L)((2i+1)h - \alpha/2)}$$

neutral dipoles situated at the even and odd sites have opposite directions. Here one may note that the formal substitution $\sigma(i) \to (-1)^i \sigma(i) + 1 - (-1)^i$ in the grand partition function converts the interaction energy between molecules to the previous form (19). However, this transformation changes the form of interaction with external fields. The $2M$-periodic reduction

$$
\begin{aligned}
a_{2jM-M} &= e^{-\pi/L(2jh - i\alpha_M)} \\
b_{2jM-M} &= e^{-\pi/L(2jh + i\alpha_M)} \\
a_{2jM-M+1} &= e^{-\pi/L(2jh - i\alpha_{M-1})}, \ldots \\
b_{2jM-M+1} &= e^{-\pi/L(2jh + i\alpha_{M-1})}, \ldots \\
a_{2jM-1} &= e^{-\pi/L(2jh - i\alpha_1)} \\
b_{2jM-1} &= e^{-\pi/L(2jh + i\alpha_1)} \\
a_{2jM} &= e^{-\pi/L((2j+1)\,h + i\alpha_1)}, \ldots \\
b_{2jM} &= -e^{\pi/L((2j+1)\,h - i\alpha_1)}, \ldots \\
a_{2jM+M-2} &= e^{-\pi/L((2j+1)\,h + i\alpha_{M-1})} \\
b_{2jM+M-2} &= -e^{-\pi/L((2j+1)\,h - i\alpha_{M-1})} \\
a_{2jM+M-1} &= e^{-\pi/L((2j+1)\,h + i\alpha_M)} \\
b_{2jM+M-1} &= -e^{-\pi/L((2j+1)\,h - i\alpha_M)}
\end{aligned}
$$

describes a gas of neutral dipoles lying on the middle line between two ideals dielectrics. Dipoles are pointed normally to the boundaries. They can switch their orientations ("up" and "down") and internal degrees of freedom are characterized by M different dipole moments α_j. In general, we have to introduce M different chemical potentials describing "internal energy" of the dipole molecule.

Another possible generalization describes mixtures of the $+2$ charge molecules with both parallel and perpendicular orientations of dipoles. Such models can describe polar plasmas where molecules can perform discrete rotations by $\pi/2$. More complicated types of mixtures of plasma particles are possible as well. Several physical variables can be calculated here. These are the number density, polarization and the pressure. Using the methods of soliton theory the τ-functions (5) can be represented as determinants of some matrices which is of great help for evaluations of the partition functions. For the self-similar cases these matrices appear to have the Toeplitz form and, as a result, they can be diagonalized by the discrete Fourier transformation. So, using the connection with Ising models the BKP-results of ref. 1 can be easily rewritten in the Coulomb gas language as well.

6. CONCLUSIONS

Constructions considered so far have one essential drawback from the point of view of statistical physics: partition functions derived from the τ-functions of integrable hierarchies have fixed temperatures. A possible way of overcoming this obstacle is to look for some quantum generalizations of classical hierarchies.

Grand partition function of two component plasma can be expressed in terms of the following equivalent field theories: Sine-Gordon, Thirring or sigma-model.[12] Lattice versions of the neutral Coulomb plasma correspond to some discretizations of these models: lattice Sine-Gordon, scalar Hubbard or XXZ model. The two component plasma at the inverse temperature $\Gamma = 2$ is mapped onto the free fermion point, e.g., of the 1D scalar Hubbard model. It can also be mapped to free spin 1/2 fermions on the lattice.[6] It is known[11] that the soliton equations can be derived in the framework of the free fermion formalism and they are equivalent to Plucker relations on the infinite dimensional Grassmanian manifold. From our point of view, the relation of Coulomb plasmas to the theory of integrable hierarchies described here sheds some new light on the relation between fermion models and different kinds of plasmas at fixed temperatures.

Variation of the temperature from the critical value, which in some models (e.g., in the 1D Ising chains picture) is not qualitatively distinguished from the other ones, would correspond to a generalization of the free fermion formalism for integrable hierarchies to the case of interacting fermions. Unfortunately, general principles of building the corresponding generalizations of the τ-function is not known. E.g., in one of the possible approaches,[15] where the quantum nonlinear Schrödinger equation is discussed, the notion of τ-function is clearly defined only in the limiting cases of the free fermion points (e.g., for the impenetrable Bose gas). It is not obvious that solutions of the corresponding integro-differential equations can be expressed in terms of a τ-function at general couplings and that such a function would make sense in the plasma picture.

Concluding this article we discuss possible physical significance of the models considered in it going beyond the Coulomb gas picture. As was shown in ref. 1 there are nice interpretations of the 1D cases from the point of view of Ising magnets. However, various boundary conditions arising within the intrinsically 2D Coulomb interaction systems look somewhat artificial in the Ising picture. Still, a number of 2D Ising models with such non-local exchange can be formulated which are exactly solvable by the techniques due to Gaudin.[2]

Another possible application concerns the fractional quantum Hall effect (FQHE). It was shown by Laughlin that the correlation functions in the appropriate state of the FQHE at the filling factor $1/m$ coincide with those of the one component Coulomb plasma for $\Gamma = 2m$. Our model is a bit different from the pure plasma case, since we have non-trivial boundary conditions. However, by placing the domain of concentration of the charged particle far from the boundary we get the Laughlin plasma. The connection with integrable hierarchies is then valid for the Laughlin states at the full filling $1/m = 1$.

ACKNOWLEDGMENTS

The authors are indebted to F. D. M. Haldane, V. I. Inozemtsev, V. E. Korepin, S. P. Novikov and V. B. Priezzhev for some remarks and stimulating discussions. The work of I.L. is supported by a fellowship from NSERC (Canada), V.S. is supported in part by the RFBR (Russia) Grant 97-01-01041 and the INTAS Grant 96-0700.

REFERENCES

1. I. Loutsenko and V. Spiridonov, *JETP Lett.* **66**:789 (1997); *Nucl. Phys. B* **538**:731 (1999).
2. M. Gaudin, *Nucl. Phys.* **85**:545 (1966); *J. Physique* **34**:511 (1973), **46**:1027 (1985).
3. A. Alastuey and J. L. Lebowitz, *J. Physique* **45**:1859 (1984).
4. B. Jancovici, *J. Stat. Phys.* **34**:803 (1984).
5. P. J. Forrester, *J. Phys. A: Math. Gen.* **18**:1419 (1985).
6. F. Cornu and B. Jancovici, *J. Chem. Phys.* **90**:2444 (1989).
7. P. J. Forrester, B. Jancovici, and G. Téllez, *J. Stat. Phys.* **84**:359 (1996).
8. B. Nienhuis, in *Phase Transitions* **11**:1 (1987).
9. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).
10. M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).
11. E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, in *Nonlinear Integrable Systems*, M. Jimbo and T. Miwa, eds. (World Scientific, Singapore, 1983), p. 41.
12. S. Samuel, *Phys. Rev. D* **18**:1916 (1978).
13. R. Hirota, *J. Math. Phys* **14**:810 (1972).
14. V. Spiridonov, in *Proc. of the Workshop on Harmonic Oscillators* (College Park, USA, March 199), NASA Conf. Publ. 3197, p. 93; *Phys. Rev. A* **52**:1909 (1995).
15. I. R. Its, A. G. Isergin, V. E. Korepin, and N. A. Slavnov, *Int. J. Mod. Phys. F* **4**:1003 (1990).